Inverse analysis of suction controlled oedometer tests

G. Sorbino, M. Calvello, S. Cuomo
Department of Civil Engineering, University of Salerno, Italy

ABSTRACT: Modelling of the groundwater regime in unsaturated conditions requires the definition of the soil water content and conductivity curves. Suction controlled oedometer test are commonly used for experimental measures of the soil water content as a function of soil suction while hydraulic conductivity is generally estimated through measured soil water content. This paper presents a numerical procedure for selecting, among different analytical models of the soil water content and conductivity curves, the ones that best fit the experimental data. This is achieved by an inverse analysis of the transient unsaturated water flow through the oedometric specimen. A comparison among the numerical results for different combinations of analytical models allows to objectively define the best pair of models and the optimal values of the relative parameters.

1 INTRODUCTION

An adequate knowledge of the unsaturated soil hydraulic properties plays a fundamental role in the analysis of many geotechnical problems where the modelling of suction regimen is required.

As it is well known, the hydraulic characterization of unsaturated soils is determined by the knowledge of the relationships between the stress state variables and both the hydraulic conductivity $k$ and the volumetric water content $\theta$ (i.e. the product of porosity $n$ and the degree of saturation $S_r$). The stress state variables are identified by the net total stress $(\sigma-u_a)$ and the matric suction $s = (u_a-u_w)$, where $\sigma =$ total stress, $u_a =$ pore air pressure and $u_w =$ pore water pressure (Matyas & Radhakrishna 1968, Fredlund & Rahardjio 1993).

However, it must be noted that unsaturated soil in the field, undergoing to processes of drying and wetting due to climatic changes, is often subjected to more frequent and significant changes in suction than in net total stress. Therefore, an appropriate estimate of the volumetric water content and hydraulic conductivity can be obtained by considering both soil-water content and hydraulic conductivity relationships for a constant value of the net total stress.

Such relationships can be determined through laboratory tests. It must be noted, however, that laboratory tests aimed to provide a direct measurement of unsaturated hydraulic conductivity function are quite time-consuming compared to those generally used for the water content function (Fredlund & Xing 1994). For this reason, usually the hydraulic conductivity function is not experimentally investigated. A common procedure consists in adopting one of the available analytical models proposed in the literature and to provide an estimation of the related unknown parameters using only the experimental data acquired on the water content.

An alternative procedure for the estimation of the unsaturated hydraulic conductivity function using only data coming from the experimental determination of the soil-water content was proposed by Sorbino (1994, 1995). Such data can derive by different apparatuses (e.g. Suction Controlled Oedometer, Volumetric Extractor, Richard Pressure Plate, and so on) that allow the determination of the soil-water content by applying variations of suction at the bottom boundary of the specimen. The procedure consists in the numerical integration of the equations governing the transient unsaturated flow processes triggered by a variation of suction. The calculated values of the hydraulic conductivity at different suction can, then, be interpreted by means of the analytical models providing the best fit between the measurements of the volume of water flowing in or out of the specimen during the transient unsaturated water flow and the numerical results.

A further improvement of the above procedure can be pursued by numerically modeling the hydraulic response of the specimen during the entire test. The unknown hydraulic properties of the specimen are defined using different analytical models for both the soil water content and conductivity curves.
Such analytical models are then calibrated by minimizing the error between the numerical results and the experimental data.

In the present paper, after illustrating the proposed procedure, an application will be shown with reference to the experimental results obtained for a natural soil tested in a Suction Controlled Oedometer (SCO). The procedure allows to quantitatively compare the effectiveness of different analytical relationships towards the modeling of both the soil water content and conductivity curves.

2 PROPOSED NUMERICAL PROCEDURE

The proposed procedure calibrates different analytical relationships of the soil conductivity and the soil water content curves in order to select the ones that best fit the experimental data coming from SCO tests.

This is achieved by the numerical modelling, at each step of the laboratory test, of the transient unsaturated water flow through the specimen integrating the well known Richards’ differential equation (Richards 1931). Figure 1 shows the scheme for the numerical model and the time-dependent boundary condition at the bottom of the specimen, which refer to the drying and wetting suction steps of the test. The initial suction distribution within the specimen is assumed constant.

The hydraulic properties of the specimen are assumed unknowns, but defined by analytical relationships. Many relationships exist in the literature for the soil conductivity and the soil water content curves, which are defined by a number of parameters (Burdine 1953, Green & Corey 1971, Mualem 1976, Fredlund et al. 1994, van Genuchten 1980, among others). They may be summarized using the following expressions that depend on the suction s and parameters \((a_1,..,a_n)\) and \((b_1,..,b_m)\):

\[
\theta = \theta(s, a_1, a_2, ..., a_n)
\]

\[
k = k(s, b_1, b_2, ..., b_m)
\]

Evidently, the results of the numerical analysis depend on: (i) which analytical relationship is used to model the soil conductivity and the soil water content curves and (ii) the assumed values of the independent parameters needed to define the curves. Both the choice of the most appropriate analytical relationships and the calibration of the model parameters is achieved by using the inverse analysis procedure shown in Figure 2.

Within this procedure, a regression analysis is performed to minimize an objective function, \(S(b)\), which quantifies the error between computed results and observations (i.e. water volumes measurements during all the steps of the laboratory test):

\[
S(b) = [y - y'(b)]^{T} \omega [y - y'(b)] = \epsilon^{T} \omega \epsilon
\]

where: \(b\) is the vector of the parameters being estimated; \(y\) is the vector of the observations being matched by the regression; \(y'(b)\) is the vector of the corresponding computed values; \(\omega\) is the weight matrix, wherein every observation’s weight is taken as the inverse of its error variance; and \(\epsilon\) is the vector of residuals.

At the end of the regression analysis, if the model fit is not “optimal”, the procedure is repeated until the model is optimized. For details on the procedure used and on how to choose the relevant parameters to optimize by inverse analysis see Calvello (2002) and Calvello and Finno (2004).
3 MATERIALS AND METHODS

The proposed procedure is tested for natural unsaturated pyroclastic soils covering a wide area (3000 km²) inside the Campania region (Southern Italy). These soils originated from the eruptive activities of Somma-Vesuvius volcano and are systematically involved in shallow landslides of flow-type triggered by rainfall (Cascini et al. 2000; Cascini 2004).

In saturated conditions, pyroclastic soils have an hydraulic conductivity ranging between $5.0 \times 10^{-6}$ m/s and $4.8 \times 10^{-5}$ m/s. In unsaturated conditions, the soil water content curves were obtained through the Suction Controlled Oedometer tests, the Volumetric Pressure Plate Extractor and the Richard Pressure Plate (Sorbino & Foresta 2002; Bilotta et al. 2005).

The proposed procedure is tested with reference to a Suction Controlled Oedometer (SCO) performed on a specimen whose main physical properties are reported in Table 1. Particularly, the test consisted in a drying and a wetting path over a suction range from 50 kPa to 250 kPa (Figure 3), under a constant vertical net stress equal to 20 kPa.

Table 1. Initial physical properties of tested specimen.

<table>
<thead>
<tr>
<th>w (%)</th>
<th>S (%)</th>
<th>n (-)</th>
<th>$\varrho$ (kN/m³)</th>
<th>$\varrho_d$ (kN/m³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>47.47</td>
<td>50.66</td>
<td>0.70</td>
<td>11.0</td>
<td>7.50</td>
</tr>
</tbody>
</table>

Figure 3. Boundary condition at the bottom of the specimen.

The main results of the SCO test are shown in Figure 4. They indicate that only negligible variation of the void ratio occur both in the drying and wetting paths (Fig. 4b), while a hysteretic behaviour is outlined in the plot of the volumetric water content versus suction (Fig. 4a).

Among the analytical models of soil water content curves available in literature, the relationship proposed by Van Genuchten (1980) is used for the variation of the volumetric water content against suction, $\theta$:

$$\theta(s) = \theta_{ws} - \frac{(\theta_{ws} - \theta_{wr})}{[1 + (\alpha s)^n]^{m}}$$  \hspace{1cm} (4)

where: $\theta_{ws}$ and $\theta_{wr}$ are the saturated and residual volumetric water content respectively, $\alpha$ is an empirical parameter (kPa$^{-1}$) whose reciprocal value can be assumed as the air entry value (a.e.v.), $n$ is a fitting constant reflecting the slope of the volumetric water content curve, and $m$ is a parameter linked to $n$ by means of the equation $m=1-1/n$.

As for the soil conductivity curves, these are derived from Eq. 4 assuming that the values of the hydraulic conductivity at a given suction level are directly related to the values of the porosity at that suction level by means of a statistical relationship employing a pore connectivity parameter (Fredlund and Xing 1994). Within this work, two different relationships between the hydraulic conductivity, $k$, and the volumetric water content, $\theta$, respectively proposed by Mualem (1976) and Burdine (1953), are used. These two relationships yield to the following analytical functions expressing the variation of hydraulic conductivity against suction:

$$k(s) = k_s \times \left[\frac{1 - (\alpha s)^{n-1}}{[1 + (\alpha s)^n]^{m/2}}\right]^{2m}$$ \hspace{1cm} (5)

$$k(s) = k_s \times \left[\frac{1 - (\alpha s)^{n-2}}{[1 + (\alpha s)^n]^{m}}\right]^{m}$$ \hspace{1cm} (6)

where: $k_s$ is the saturated hydraulic conductivity and $\alpha, n, m$ are the same parameters used for the volumetric water content curve in Eq. 4.
4 NUMERICAL ANALYSIS

The selection of the soil water content and conductivity curves is firstly based on the numerical simulation of the transient pore water pressures observed during the SCO test. This is achieved by a numerical analysis carried out using the Finite Element code SEEP/W (GeoSlope 2004), which is able to integrate the Richards’ differential equation. The domain is discretised using quadrilateral elements (5×10⁻⁴ m tall) as shown in Figure 5. The time-dependent bottom boundary conditions refer to the suction values imposed during drying and wetting steps of Figure 4. Within each suction step, which lasts 24 hours, the transient numerical analysis is carried out with incremental time steps, starting with 1 minute and following a geometric progression. The initial suction distribution within the specimen is assumed equal to 50 kPa.

![Figure 5. Finite Element model of the specimen during the SCO test.](image)

Table 2 shows the first estimation of the parameters of the soil water content and conductivity curves based on the procedure proposed by Sorbino (1994, 1995). A comparison among the experimental data and the numerical results is shown in Figure 6, where the curves VG_M_S94 and VG_S_S94 respectively indicate that equations 3 and 4 (i.e. van Genuchten and Mualem models) and equations 3 and 5 (i.e. van Genuchten and Burdine models) are used in the analysis. With these values of parameters the Mualem model seems to perform better than the Burdine model, yet even in the first case the fit with the experimental data is not optimal.

![Figure 6. Comparison between the measured volumes of water, flowing in and out of the specimen during the wetting and drying paths of the SCO test, and the numerical results based on the parameters’ values reported in table 2.](image)

![Figure 7. Comparison of the soil water content curves obtained with the parameters’ values reported in tables 2 and 3.](image)

Table 2. Estimation of the parameters of the θ(s) and k(s) curves based on the numerical procedure S94 (Sorbino, 1994).

<table>
<thead>
<tr>
<th>ID</th>
<th>θws</th>
<th>θwr</th>
<th>α</th>
<th>n</th>
<th>k</th>
<th>(m/min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S_94</td>
<td>0.66</td>
<td>4.3×10⁻⁷</td>
<td>0.19</td>
<td>1.28</td>
<td>3.0×10⁴</td>
<td></td>
</tr>
</tbody>
</table>

In order to improve the fit between the experimental data and the numerical results, the procedure described in Section 2 is used. The unknown parameters of the soil conductivity and the soil water content curves are calibrated either using all the available measures or those referring only to the drying and wetting paths. Table 3 shows the optimal values of the parameters for the two combinations of analytical curves and for the three sets of observations used. Figures 7 and 8 shows the comparison between the 6 sets of optimized curves and their initial estimates. The results of the 6 models are shown in Figures 9, 10 and 11.
The results of the analysis indicate that both the combinations of analytical curves (i.e. VG_M and VG_B), once calibrated, are capable of adequately reproducing the experimental behavior of the specimen. However, the values of the calibrated models’ parameters are, in the two cases, very different. And they also significantly differ according to whether all observations (i.e. _a) or only some of the observations (i.e. _d and _w) are used in the regression.

In order to move from a qualitative comparison of the results to a comparison based on quantitative measures of the model fit, the value of the objective function $S(b)$ from equation (3) can be employed to derive two useful statistics of the analysis: the model error variance, $s^2$, and the model fit improvement, $FI$.

$$s^2 = S(b)/ND$$  \hspace{1cm} (7)

$$FI = \frac{S(b)_{\text{initial}} - S(b)_{\text{optimized}}}{S(b)_{\text{initial}}}$$  \hspace{1cm} (8)

where: $ND$ is the number of observations, $S(b)_{\text{initial}}$ is the initial value of the objective function, $S(b)_{\text{optimized}}$ is the value of the objective function for the optimized set of parameters.
The model error variance, $s^2$, is a measure of the consistency between the fit achieved by the calibrated model and the accuracy of the data as reflected in the weighting. The model fit improvement, $FI$, quantifies the effectiveness of the proposed numerical procedure in improving the first calibration of the model (i.e. the first estimate of the parameters).

Figure 12 compares the error variances of the 8 models considered. The lower the values of $s^2$, the better the fit between the observations and the numerical results. The comparison clearly show that the best simulations are the ones calibrated on the basis of the drying or wetting paths only, thus indicating that the hysteretic behavior of the specimen does have a significant effect on the model. On the contrary, only minor differences exist between two equivalent models based on the VG_M and VG_B sets of curves, thus confirming that both assumptions are adequate.

![Figure 12. Error variances of the 8 models (lower values = better fitting).](analytical_curves)

Figure 13 compares the $FI$ values of the 6 models calibrated using the proposed procedure. These values, which indicate by what percentage the optimized results improved compared to the initial fit between experimental data and computed results, are always greater than 80%, thus showing that all the optimized models are significantly better that the initial one.

![Figure 13. Performance of the inverse analysis for the 6 optimized models (improvement from initial models VG_M_S94 and VG_B_S94).](analytical_curves)

5 CONCLUSIONS

This paper presented a procedure for selecting and calibrating different analytical models of the soil water content and conductivity curves on the basis of the best fit between experimental data from a Suction Controlled Oedometer test and the results of a numerical simulation of the test. Both the comparison and the calibration of the models are based on an inverse analysis of the transient unsaturated water flow through the tested specimen.

Two sets of analytical relationships were used of the soil water content and conductivity curves. The numerical results showed that both sets are capable of adequately reproducing the experimental behavior of the specimen once the parameters of both curves are appropriately calibrated. The best simulations are the ones calibrated on the basis of the drying or wetting paths only, thus indicating that the hysteretic behavior of the specimen does have a significant effect on the model. The results of the analysis also clearly showed the effectiveness of the proposed numerical procedure in calibrating the analytical model parameters and significantly improving the initial fit between experimental data and computed results.

6 REFERENCES


